Analysis of interfacial crack by means of hypersingular integro-differential equations

Nao-Aki Noda*, & Yu Zhang*

*Mechanical Engineering Department, Kyushu Institute of Technology, Kitakyushu 804-8550, Japan

Abstract

Numerical solutions of hypersingular integro-differential equations are discussed in the analysis of interfacial crack in two and three dimensional bimaterials subjected to general internal pressure. The problem is formulated on the basis of the body force method. In the numerical analysis, unknown body force densities are approximated by the products of the fundamental density functions and power series, where the fundamental density functions are chosen to express singular behavior along the crack front of the interface crack exactly. The present method gives rapidly converging numerical results and highly satisfied boundary conditions throughout the crack boundary. The stress intensity factors are given with varying the material combination and aspect ratio of the crack. It is found that the stress intensity factors K_I and K_{II} are determined by the bimaterials constant ε alone, independent of elastic modulus ratio and Poisson's ratio.

Keywords: hypersingualr intergro-differential equations; body force method; stress intensity factors

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Introduction

In recent years, composite materials and adhesive or bonded joints are being used in wide range of engineering field. The fracture of composites and bonded dissimilar materials is induced mainly from the interfacial region because the angular corner of bonded materials induces singular stress and crack initiation at the interface. Particular flaws or cracks lying along the interface reduce the strength of the structure significantly. Hence, analysis of interfacial cracks in dissimilar materials is very important from the view point of interfacial strength.

In the present paper, the two-dimensional interfacial crack subjected to bending stress, and the three-dimensional interfacial rectangular crack subjected to tension(Fig.1) will be analyzed on the basis of integro-differential equations.

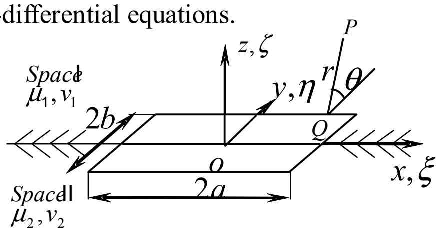

Fig.1. Interfacial rectangular crack subjected to general internal pressure $p_x(x,y)$, $p_y(x,y)$ and $p_z(x,y)$.

Hypersingular integro-differential equations and numerical solutions

Consider two dissimilar elastic half-spaces bonded together along the *x–y* plane with a fixed rectangular Cartesian coordinate system x_i ($i=x,y,z$). Suppose that the upper half-space is occupied by an elastic medium with constants $(\mu_1,\ \nu_1)$, and the lower half-space by an elastic medium with constants $(\mu_2,\ \nu_2)$. Here, μ_1,μ_2 are shear modulus for space 1 and space 2, and ν_1,ν_2 are Poisson's ratio for space 1 and space 2. The crack is assumed to be located at the bimaterial interface.

Hypersingular intergro-differential equations for two dimensional cracks when $a/b\to\infty$ on a bimateral interface are shown in Eq(1)[1].

$$\beta C\frac{\partial\Delta u_z}{\partial y}+\frac{C}{\pi}\int_{-b}^{b}\frac{\Delta u_y}{(y-\xi)^2}d\xi=-p_y$$
$$-\beta C\frac{\partial\Delta u_y}{\partial y}+\frac{C}{\pi}\int_{-b}^{b}\frac{\Delta u_z}{(y-\xi)^2}d\xi=-p_z \qquad (1a)$$

$$C=\frac{2\mu_1(1+\alpha)}{(1-\beta^2)(\kappa_1+1)}=\frac{2\mu_2(1-\alpha)}{(1-\beta^2)(\kappa_2+1)}$$

$$\alpha=\frac{\mu_2(\kappa_1+1)-\mu_1(\kappa_2+1)}{\mu_2(\kappa_1+1)+\mu_1(\kappa_2+1)}$$
$$\beta=\frac{\mu_2(\kappa_1-1)-\mu_1(\kappa_2-1)}{\mu_2(\kappa_1+1)+\mu_1(\kappa_2+1)} \qquad (1b)$$

$$\kappa_l=\begin{cases}\dfrac{3-\nu_l}{1+\nu_l} & \text{plane stress}\\ 3-4\nu_l & \text{plane strain}\end{cases} \qquad (1c)$$

$$\Delta u_i(y)=u_i(y,0^+)-u_i(y,0^-)(i=y,z) \qquad (1d)$$

Here $\Delta u_x,\Delta u_z$ are the crack opening displacements.

Hypersingular intergro-differential equations for three dimensional cracks on a bimateral interface were derived by Chen–Noda-Tang (1999) and expressed as shown in Eq.(2)[2].

$$\mu_1(\Lambda_2-\Lambda_1)\frac{\partial\Delta u_z(x,y)}{\partial x}+\mu_1\frac{(2\Lambda-\Lambda_1-\Lambda_2)}{2\pi}\times\int_s\frac{1}{r^3}\Delta u_x(\xi,\eta)dS(\xi,\eta)+3\mu_1\frac{(\Lambda_1+\Lambda_2-\Lambda)}{2\pi}\times\int_s\frac{(x-\xi)^2}{r^5}\Delta u_x(\xi,\eta)dS(\xi,\eta)+3\mu_1\frac{(\Lambda_1+\Lambda_2-\Lambda)}{2\pi}\int_s\frac{(x-\xi)(y-\eta)}{r^5}\times\Delta u_y(\xi,\eta)dS(\xi,\eta)=-p_x(x,y) \qquad (2a)$$

$$\mu_1(\Lambda_2-\Lambda_1)\frac{\partial\Delta u_z(x,y)}{\partial y}+\mu_1\frac{(2\Lambda-\Lambda_1-\Lambda_2)}{2\pi}\times\int_s\frac{1}{r^3}\Delta u_y(\xi,\eta)dS(\xi,\eta)+3\mu_1\frac{(\Lambda_1+\Lambda_2-\Lambda)}{2\pi}\times\int_s\frac{(x-\xi)(y-\eta)}{r^5}\Delta u_x(\xi,\eta)dS(\xi,\eta)+3\mu_1\frac{(\Lambda_1+\Lambda_2-\Lambda)}{2\pi}\int_s\frac{(y-\eta)^2}{r^5}\times\Delta u_y(\xi,\eta)dS(\xi,\eta)=-p_y(x,y) \qquad (2b)$$

$$\mu_1(\Lambda_1-\Lambda_2)\left(\frac{\partial\Delta u_x(x,y)}{\partial x}+\frac{\partial\Delta u_y(x,y)}{\partial y}\right)+\mu_1\frac{(\Lambda_1+\Lambda_2)}{2\pi}\int_s\frac{1}{r^3}\Delta u_z(\xi,\eta)dS(\xi,\eta)=-p_z(x,y) \qquad (2c)$$

$$\Lambda=\frac{\mu_2}{\mu_1+\mu_2},\ \Lambda_1=\frac{\mu_2}{\mu_1+\kappa_1\mu_2},\ \Lambda_2=\frac{\mu_2}{\mu_2+\kappa_2\mu_1},$$
$$\kappa_1=3-4\nu_1,\ \kappa_2=3-4\nu_2,\ r^2=(x-\xi)^2+(y-\eta)^2 \qquad (2d)$$
$$(x,y)\in S, S=\{(x,y)\,||\,x|\le a,|\,y|\le b\}$$

$$\Delta u_i(x)=u_i(x,y,0^+)-u_i(x,y0^-)(i=x,y,z) \qquad (2e)$$

In Eq.(2), unknown functions are crack opening displacements $\Delta u_x(x,y), \Delta u_y(x,y), \Delta u_z(x,y)$, Here, (ξ,η,ζ) is a rectangular coordinate (x,y,z) where the displacement discontinuities are distributed, the notations p_x , p_y , p_z denote surface tractions in the x,y,z directions at the crack surface. Since the integral has a hypersingularity of the form r^{-3} when $x=\xi$ and $y=\eta$, the integration should be interpreted in a sense of a finite part integral in the region *S*.

Numerical solutions for Eq.(2)

In the present analysis, the fundamental densities and polynomials have been used to approximate the unknown functions as continuous functions. First, we put

$$\begin{aligned}\Delta u_x(\xi,\eta) &= w_x(\xi,\eta)F_x(\xi,\eta),\\ \Delta u_y(\xi,\eta) &= w_y(\xi,\eta)F_y(\xi,\eta),\\ \Delta u_z(\xi,\eta) &= w_z(\xi,\eta)F_z(\xi,\eta).\end{aligned} \quad (3)$$

The fundamental densities $w(\zeta,\eta)$ lead to express the oscillation stress singularity and overlapping of crack surfaces along the crack front exactly.

$$w_x(\xi,\eta)=\sum_{l=1}^{2}\frac{1+k_l}{4\mu_l\cosh\pi\varepsilon}\sqrt{a^2-\xi^2}\sqrt{b^2-\eta^2}\,\sin\left(\varepsilon\ln\left(\frac{a-\xi}{a+\xi}\right)\right),$$

$$w_y(\xi,\eta)=\sum_{l=1}^{2}\frac{1+k_l}{4\mu_l\cosh\pi\varepsilon}\sqrt{a^2-\xi^2}\sqrt{b^2-\eta^2}\,\sin\left(\varepsilon\ln\left(\frac{b-\eta}{b+\eta}\right)\right),$$

$$w_z(\xi,\eta)=\sum_{l=1}^{2}\frac{1+k_l}{4\mu_l\cosh\pi\varepsilon}\sqrt{a^2-\xi^2}\sqrt{b^2-\eta^2}\,\cos\left(\varepsilon\ln\left(\frac{a-\xi}{a+\xi}\right)\right)\times\cos\left(\varepsilon\ln\left(\frac{b-\eta}{b+\eta}\right)\right). \quad (4)$$

To satisfy the boundary conditions for the rectangle region of the interfacial crack, $F_i(\xi,\eta)(i=x,y,z)$ can be approximated by polynomials, for example:

$$\begin{aligned}F_x(\xi,\eta)&=\alpha_0+\alpha_1\eta+\cdots+\alpha_{n-1}\eta^{(n-1)}+\alpha_n\eta^n+\alpha_{n+1}\xi\\ &+\alpha_{n+2}\xi\eta+\cdots+\alpha_{2n}\xi\eta^n+\cdots+\alpha_{l-n-1}\xi^m+\alpha_{l-n}\xi^m\eta\\ &+\cdots+\alpha_{l-1}\xi^m\eta^n=\sum_{i=0}^{l-1}\alpha_i G_i(\xi,\eta),\end{aligned} \quad (5)$$

$$l=(m+1)(n+1),$$

$$G_0(\xi,\eta)=1,\ G_1(\xi,\eta)=\eta,\ \cdots, G_{n+1}(\xi,\eta)=\xi,$$

$$\cdots, G_{l-1}(\xi,\eta)=\xi^m\eta^n.$$

Results and Discussions

In Fig.1,the stress intensity factors K_I, K_{II} and the dimensionless stress intensity factors F_I, F_{II} are defined as the following.

$$K(Q)=K_I(Q)+iK_{II}(Q)=\lim_{r\to 0}\sqrt{2\pi r}^{\,1/2-i\varepsilon}[\sigma_z(r,\theta)+i\tau_{zx}(r,\theta)]_{\theta=0}$$

$$F_I+iF_{II}=\frac{K_I(x,y)\big|_{y=\pm b}+iK_{II}(x,y)\big|_{y=\pm b}}{\sigma_z^{\infty}\sqrt{\pi b}} \quad (6)$$

$$=\sqrt{a^2-x^2}\left(\cos\left(\varepsilon\ln(\frac{a-x}{a+x})\right)F_z(x,y)\big|_{y=\pm b}+2i\varepsilon F_y(x,y)\big|_{y=\pm b}\right)$$

Table 1 shows the dimensionless stress intensity factors F_I and F_{II} when $a/b\to\infty$ under general internal pressure p_0 , $p_0(y/a)^2$ and $p_0(y/a)^3$. These results are obtained by applying a similar numerical solution described above. For constant pressure $p_z(y)=p_0$, the exact solution is obtained at $M=1$. For $p_z(y)=p_0(y/a)^2$, the exact solution is obtained at $M=3$. Generally for $p_z(y)=p_0(y/a)^n$, the exact solution is obtained at $M=n+1$.

Table 2 shows the dimensionless stress intensity factors F_I and F_{II} at the point $(x,y)=(0,b)$ in Fig.1. It is found that dimensionless stress intensity factors F_I and F_{II} are constant for the variation of the shear modulus ratio μ_2/μ_1 and Poisson's ratio ν_1 , $\nu_2=0$ ~ 0.5 if ε is constant. In other words, the stress intensity factors K_I and K_{II} of planer interface cracks in bimaterials are determined by the bimaterial constant ε alone, independent of the shear modulus ratio and Poisson's ratio, and of course, Young's modulus ratio[3].

Table 1 Dimensionless stress intensity factors F_I and F_{II} at $\varepsilon=0.02$ when $a/b\to\infty$ with M.

	$p_z(y)=p_0$		$p_z(y)=p_0\left(\frac{y}{a}\right)^2$		$p_z(y)=p_0\left(\frac{y}{a}\right)^3$	
M	F_I	F_{II}	F_I	F_{II}	F_I	F_{II}
1	1.0000	0.0400	0.0000	0.0000	0.0000	0.0000
2	1.0000	0.0400	0.2500	0.0010	0.1248	0.0010
3	1.0000	0.0400	0.4992	0.0333	0.2219	0.0178
4	1.0000	0.0400	0.4992	0.0333	0.3741	0.0333
5	1.0000	0.0400	0.4992	0.0333	0.3741	0.0333
6	1.0000	0.0400	0.4992	0.0333	0.3741	0.0333
7	1.0000	0.0400	0.4992	0.0333	0.3741	0.0333
8	1.0000	0.0400	0.4992	0.0333	0.3741	0.0333

Table 2 Dimensionless stress intensity factors F_I and F_{II} at the point $(x,y)=(0,b)$ in Fig. 1 under constant pressure p_0.

	F_I				F_{II}			
ε	a/b=1	a/b=2	a/b=4	a/b=8	a/b=1	a/b=2	a/b=4	a/b=8
0.02	0.7528	0.9052	0.9760	0.9947	0.0274	0.0352	0.0388	0.0397
0.04	0.7509	0.9038	0.9750	0.9938	0.0542	0.0696	0.0768	0.0786
0.06	0.7478	0.9013	0.9730	0.9920	0.0799	0.1027	0.1134	0.1160
0.08	0.7433	0.8975	0.9699	0.9891	0.1040	0.1338	0.1479	0.1515
0.10	0.7373	0.8921	0.9654	0.9848	0.1263	0.1627	0.1801	0.1845

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